Some Simple Algorithms[1] for Calculating Derivatives

The Derivative of a Constant is Zero

Suppose we are told that

$$x = x_o$$

where x_o is a constant and x represents the position of an object on a straight line path, in other words, the distance that the object is in front of a start line. The derivative of x with respect to t

$$\frac{dx}{dt} \text{ which can also be written } \frac{d}{dt}x$$

is then just the derivative of x_o with respect to t.

$$\frac{dx}{dt} = \frac{dx_o}{dt}$$

Now according to the mathematicians, the derivative of a constant is zero, so we have:

$$\frac{dx}{dt} = 0$$

Does this make sense? This whole discussion about derivatives is relevant to the study of motion because the velocity of an object is the derivative of its position with respect to time:

$$v = \frac{dx}{dt}$$

So what we are saying now is that if $x = x_o$ (where x_o is a constant) then $v = 0$. Well $x = x_o$ means that the position of the object is not changing. So if we are talking about a car, for instance, then we must be talking about a parked car, and YES it does make sense for $\frac{dx}{dt} = 0$, that is for $v = 0$, because the velocity of a parked car is indeed zero.

[1] An algorithm is a sequence of steps. When you learned how to do long division, you were learning to execute an algorithm. The term is usually used in computer science where the goal is often to write an algorithm in a language that a computer can understand. But the term is not limited to computer science. In fact, you have been executing algorithms yourself ever since you first learned that one plus one is two.

The Derivative of *t* With Respect to *t* Is 1.

Consider the function[2]

$$f(t) = t$$

which is to be read "*f* of *t* equals *t*". According to the mathematicians, if $f = t$ then

$$\frac{df}{dt} = \frac{d}{dt}t = \frac{dt}{dt} = 1$$

Does this make sense? When you take the derivative of something with respect to time, you are supposed to get the rate of change of that something. If $f = t$, then $\frac{df}{dt}$ is the rate of change of the stopwatch reading. Our result, 1, with no units, can be interpreted as $1\frac{\text{s}}{\text{s}}$ or $1\frac{\text{minute}}{\text{minute}}$ or $1\frac{\text{hour}}{\text{hour}}$, any one of which is indeed the rate at which the stopwatch reading changes.

The Derivative of a Constant Times a Function is the Constant Times the Derivative of the Function.

The derivative of $x(t) = v_o f(t)$ which is to be read "*x* of *t* equals v-sub-o times *f* of *t*") where v_o is a constant is

$$\frac{df}{dt} = \frac{d}{dt}(v_o t) = v_o \frac{d}{dt}t = v_o \frac{dt}{dt}$$

but we already established that the derivative of *t* with respect to *t* is 1. So,

$$\frac{dx}{dt} = \mathrm{v}_o$$

That's a mathematical result. Now lets look at the physics[3].

If $x(t) = v_o t$ with v_o being constant, then:

(a) $x = 0$ when $t = 0$ meaning the object is "at the start line" at time zero (the instant when "the stopwatch" is started).

[2] I chose to use a function name other than *x* here because I am using *x* for position and *t* for stopwatch reading, and, the units do not allow us to set *x* equal to *t*.

[3] By "the physics" we mean the way in which the particular aspect of nature under consideration "is" or "behaves".

(b) Note that if we solve this for v_o we get $v_o = \frac{x}{t}$ which can be written as

$v_o = \frac{x-0}{t-0} = \frac{\Delta x}{\Delta t}$ which is our expression for the average velocity. Since it was stipulated that v_o is a constant, we must get the same value for average velocity no matter what (x, t) pair we pick. This means that the velocity is a constant at the value of v_o. So when $x(t) = v_o t$ the object is moving at constant velocity v_o. Velocity is the rate of change of position but that is exactly what we mean by $\frac{dx}{dt}$ so our mathematical result that $\frac{dx}{dt} = v_o$ is in agreement with the physics.

Getting back to the idea that the derivative of "a constant times a function of *t*" with respect to *t* is just the constant times the derivative of the function, lets look at another example. Suppose

$$x(t) = \frac{1}{2}at^2$$

where *a* is a constant. If *a* is a constant then $\frac{1}{2}a$ is, of course, a constant so $x(t)$ has the form

$$x(t) = \text{constant} \cdot f(t)$$

(which is to be read "*x* of *t* equals a constant times *f* of *t*") where $f(t) = t^2$. According to the rule:

$$\frac{dx}{dt} = \text{constant} \cdot \frac{d}{dt} f$$

and, in the case at hand, where the constant is $\frac{1}{2}a$ and $f = t^2$, that is, $x(t) = \frac{1}{2}at^2$, we have

$$\frac{dx}{dt} = \frac{1}{2}a\frac{d}{dt}t^2$$

which involves taking the derivative of a power of *t*.

The Derivative of t^n With Respect to t is nt^{n-1}.

Another calculus result that the mathematicians have provided us with is the fact that if

$$f(t) = t^n$$

with n being a constant then

$$\frac{df}{dt} = nt^{n-1}.$$

In other words, if you are taking the derivative, with respect to t, of t raised to a power, then all you have to do is copy the power down "out front" and reduce the power by 1. The result is thus, the value of the original power times t raised to "the original power minus 1".

Remember that example from the last section in which $x(t) = \frac{1}{2}at^2$ and we got as far as

$$\frac{dx}{dt} = \frac{1}{2}a\frac{d}{dt}t^2$$

Well, now you know how to do that last part. $\frac{d}{dt}t^2 = 2t^{2-1} = 2t^1 = 2t$ so,

$$\frac{dx}{dt} = \frac{1}{2}a2t$$

$$\frac{dx}{dt} = at \qquad (1)$$

Okay, here we go again. The derivative of anything with respect to time is the rate of change of that anything. The quantity x is position, so $\frac{dx}{dt}$ is the rate of change of position. But the rate of change of position is, by definition, the velocity. So $\frac{dx}{dt}$ is v. Thus equation 1 above is saying that

$$v = at.$$

Recall that a was stipulated to be a constant. Now if the velocity is a constant times the stopwatch reading, then that constant must be the constant rate of change of the velocity. The rate of change of velocity is, by definition, the acceleration of the object, so, when $x(t) = \frac{1}{2}at^2$ with a being a constant, we are dealing with a situation in which we have a constant value of acceleration equal to the constant a.

The Derivative of a Sum of Terms is the Sum of the Derivatives.

This is the distributive rule for the derivative operator. Suppose we have some function $x(t)$ which can be written as the sum of three other functions of t.

$$x(t) = f(t) + g(t) + h(t)$$

The rule above is just saying that

$$\frac{dx}{dt} = \frac{d}{dt}\left[f(t) + g(t) + h(t)\right]$$

$$\frac{dx}{dt} = \left[\frac{d}{dt}f(t) + \frac{d}{dt}g(t) + \frac{d}{dt}h(t)\right]$$

$$\frac{dx}{dt} = \left[\frac{df}{dt} + \frac{dg}{dt} + \frac{dh}{dt}\right]$$

Suppose for example that

$$x(t) = x_o + v_o\, t + \frac{1}{2}at^2$$

with x_o, v_o, and a being constants. Then,

$$\frac{dx}{dt} = \frac{d}{dt}\left[x_o + v_o\, t + \frac{1}{2}at^2\right]$$

$$\frac{dx}{dt} = \frac{d}{dt}x_o + \frac{d}{dt}v_o\, t + \frac{d}{dt}\frac{1}{2}at^2$$

$$\frac{dx}{dt} = 0 + v_o\ \frac{d}{dt}t + \frac{1}{2}a\frac{d}{dt}t^2$$

$$\frac{dx}{dt} = v_o\ 1 + \frac{1}{2}a\ 2t$$

$$\frac{dx}{dt} = v_o\ + a\ t$$

and since $\frac{dx}{dt}$, the rate of change of position, is just the velocity v, we have

$$v = v_o + \ a\ t$$